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A FEW REMARKS ON ROWBOTTOM CARDINALS

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ABSTRACT

It is proved that every regular Rowbottom cardinal which is greater than the continuum is strongly inaccessible. We notice that the theory "ZFC + every cardinal of cofinality ω is Rowbottom" is inconsistent. This answers a question raised by C. C. Chang and H. J. Keisler.

1. Introduction

A cardinal $\kappa > \omega_1$ is called a Rowbottom cardinal if, whenever $\mathfrak{A} = \langle A, Q, \cdots \rangle$ is a structure of countable length with $|A| = \kappa$, $Q \subset A$ and $|Q| < \kappa$, then $\mathfrak{A}$ has an elementary substructure $\mathfrak{B} = \langle B, Q \cap B, \cdots \rangle$ with $|B| = \kappa$ and $|Q \cap B| \leqq \omega$. Even without the axiom of choice this model theoretic definition is equivalent to the well-known combinatorial one [2]: $\kappa > \omega_1$ is Rowbottom iff for any $f : [\kappa]^{<\omega} \rightarrow \lambda$, where $\lambda < \kappa$, there exists a set $X \subseteq \kappa$ such that $|X| = \kappa$ and $|f''[X]^{<\omega}| \leqq \omega$. It is also well known that if κ is a Rowbottom cardinal, then either κ is weakly inaccessible or $\operatorname{cf} \kappa = \omega$.

The purpose of this paper is, using ideas of K. Prikry [3], to establish that a regular Rowbottom cardinal $\kappa > 2^\omega$ is strongly inaccessible. We indicate also some cardinals of cofinality ω which are not Rowbottom. This answers a question raised in [1].

Our notation is standard, but we do mention the following: the letters κ, λ always denote infinite cardinals. If x is a set, $|x|$ is its cardinality and $[x]^{<\omega}$ is the set of all finite subsets of x. If f is a function, then $f''x$ is the image of x under f and $f \mid x$ is the restriction of f to x. ${}^y x$ denotes the set of all functions from x into y. V_κ is the collection of all sets of rank less than κ.

In this paper all undefined set theoretical notions are taken from [2].

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2. Basic lemmas

Let us recall that two functions f, g on λ are almost disjoint if there is $\gamma < \lambda$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geqq \gamma$.

The main theorem of this paper is based on the following

LEMMA 1. *Let κ be a Rowbottom cardinal. Then*

(i) *Every family F of almost disjoint functions $f:\lambda \to \nu$, where $\nu, < \kappa$ and $\operatorname{cf}\lambda > \omega$, has less than κ elements.*

(ii) *Every family F of almost disjoint functions $f:\lambda \to \kappa$, where $\lambda < \kappa$ and $\operatorname{cf}\lambda > \omega$, has at most κ members unless κ is singular.*

PROOF. (i) Suppose, towards a contradiction, that $|F| \geqq \kappa$.

Let us define the partition $P:[F]^2 \to \lambda$ by $P(\{f, g\}) =$ least γ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha \geqq \gamma$. By the combinatorial property of κ, there exists a subfamily $G \subseteq F$ of size κ such that the image $P''[G]^2$ is at most countable. We put $\eta = \sup P''[G]^2$. As $\operatorname{cf}\lambda > \omega$, we have $\eta < \lambda$. But then $f(\eta) \neq g(\eta)$ whenever $f, g \in G$, which is absurd since $|G| = \kappa$ and $f(\eta) < \nu$ for all $f \in G$.

(ii) If $|F| > \kappa$ then choose a subfamily $H \subset F$ and an ordinal $\nu < \kappa$ so that $|H| = \kappa$ and $f \in {}^{\lambda}\nu$ for all $f \in H$ (because every function $f:\lambda \to \kappa$ is bounded) and apply (i) to get a contradiction. ■

We obtain an interesting and useful result by using a kind of device of F. Rowbottom.

LEMMA 2. *If $\kappa > 2^{\omega}$ is a Rowbottom cardinal then $\lambda^{\omega} < \kappa$ for all $\lambda < \kappa$.*

PROOF. It suffices to show the following combinatorial claim: whenever $f:[\kappa]^{<\omega} \to \lambda^{\omega}$ where $\lambda < \kappa$, there exists a subset $X \subseteq \kappa$ such that $|X| = \kappa$ and $|f''[X]^{<\omega}| \leqq \omega$.

So let $f:[\kappa]^{<\omega} \to {}^{\omega}\lambda$ and $\lambda < \kappa$. Define $g:[\kappa]^{<\omega} \to \lambda$ as follows: Suppose $x_1 < \cdots < x_n$ in κ, $n \geqq 1$ and $n = 2^i(2j+1)$; then $g(\{x_1, \cdots, x_n\}) = f(\{x_1, \cdots, x_i\})(j)$.

Now by the combinatorial characterization of κ we find a subset $Y \subseteq \kappa$ of size κ such that the image A of $[Y]^{<\omega}$ under g is at most countable. It is easy to notice that $f''[Y]^{<\omega}$ is included in ${}^{\omega}A$. Since $2^{\omega} < \kappa$ it follows immediately from the definition of κ used once more that there is an $X \subseteq Y$ which has the desired properties. ■

3. Main propositions

THEOREM 1. (i) *If $\kappa > 2^{\omega}$ is a regular Rowbottom cardinal then $2^{\lambda} < \kappa$ for all $\lambda < \kappa$.*

(ii) *If* 2^ω *is Rowbottom then* $2^\lambda = 2^\omega$ *for all* $\lambda < 2^\omega$.

PROOF. (i) Argue by contradiction. Assume that $\lambda < \kappa$ is the least cardinal such that $2^\lambda \geqq \kappa$. Clearly, $\lambda > \omega$.

At first we wish to show $\mathrm{cf}\,\lambda > \omega$. Suppose not. Our assumption is that $\lambda = \Sigma_{n<\omega}\lambda_n$ for some increasing sequence $\langle \lambda_n : n < \omega\rangle$ of cardinals. Therefore $2^\lambda = \Pi_{n<\omega} 2^{\lambda_n} = (\sup_{n<\omega} 2^{\lambda_n})^\omega$ and $\kappa = \sup_{n<\omega} 2^{\lambda_n}$ by Lemma 2. But this is absurd unless $\mathrm{cf}\,\kappa = \omega$.

At present we assign for each $f \in {}^\lambda 2$ the sequence $\bar{f} = \langle f \mid \alpha : \alpha < \lambda\rangle$. The collection $\{\bar{f} : f \in {}^\lambda 2\}$ yields a family F of 2^λ almost disjoint functions from λ into a set of cardinal less than κ. However, following Lemma 1 the family F must have less than κ elements. Having arrived at this contradiction we have thus established (i).

(ii) Now we get (ii) by simple suitable repetition of the previous proof and an application of Lemma 1 (ii). ■

REMARK. From the above proof we obtain that if $\kappa > 2^\omega$ is a Rowbottom cardinal, then either κ is a strong limit or $\mathrm{cf}\,\kappa = \omega$ and there exists $\lambda < \kappa$ of cofinality ω such that $2^{<\lambda} = \kappa$ and $2^\lambda = \kappa^\omega$. In particular, if $2^\omega < \omega_\omega$ and ω_ω is Rowbottom then $2^{\omega_n} < \omega_\omega$ for all $n < \omega$. ■

Following a remark of M. Magidor let me state the following

THEOREM 2. *Let* κ *be a Rowbottom cardinal. If* $\varphi(x)$ *is any property which can be expressed by a downward absolute formula which is absolute between* V *and* V_κ *such that* $\varphi(\alpha)$ *holds in* V *for some uncountable* $\alpha < \kappa$, *then the set* $A = \{\beta < \kappa : \varphi(\beta)\}$ *is unbounded in* κ *and either* $|A| = \kappa$ *or* $|A| = \omega$.

PROOF. We shall only show that the set A is unbounded in κ. The method of the proof is suitable to receive the second part.

We suppose the opposite and derive a contradiction. Let $\eta = \sup A$. Our assumption means that $\eta < \kappa$.

We start with the structure $\langle V_\kappa, \in\rangle$. In case of need we apply the downward Lowenheim–Skolem theorem to get its elementary substructure $\mathfrak{A}$ of size κ such that the universe B of $\mathfrak{A}$ contains κ.

Consider the structure $\mathfrak{B} = \langle B, \in, \eta, g, \eta\rangle$ which has the distinguished unary relation η, the one-to-one function $g : B \to \kappa$ and the individual constant η. By the model theoretic property of κ we find an elementary substructure $\mathfrak{C} = \langle C, \in, \eta \cap C, g \mid C, \eta\rangle$ of $\mathfrak{B}$ with $|C| = \kappa$ and $|\eta \cap C| = \omega$.

Now $\mathfrak{C}$ is a well-founded model of the axiom of extensionality and so by

Mostowski's isomorphism theorem we have the unique collapsing isomorphism π taking $\mathfrak{C}$ to a transitive structure

$$\mathfrak{M} = \langle M, \in, \pi''(\eta \cap C), \pi \circ g \mid C \circ \pi^{-1}, \pi(\eta) \rangle.$$

Put $\gamma = \pi(\eta)$. It is not difficult to see that $\pi''(\eta \cap C) = \gamma$ and so $\gamma < \omega_1$. It is also easy to prove with the use of g that $\pi''(\kappa \cap C) = \kappa$. Thus $\kappa \subseteq M$.

Finally, each ordinal $\beta < \kappa$ such that $\varphi(\beta)$ holds in $\mathfrak{M}$ is not greater than γ. Nevertheless $\alpha \in M$ and $\varphi(\alpha)$ holds in $\mathfrak{M}$ because the property φ is described by a downward absolute formula, contradicting $\gamma < \omega_1$. ∎

COROLLARY 1. (i) *If $\kappa > \omega_\omega$ is a Rowbottom cardinal then κ is the limit of the sequence of all limit cardinals less than κ. In particular, $\omega_{\omega+\omega}$ is not Rowbottom.*

(ii) *If $\kappa > \omega_{\omega_1}$ is a Rowbottom cardinal then κ is a fixed point of the aleph function.*

(iii) *The theory* "ZFC + *every cardinal of cofinality ω is Rowbottom*" *is not consistent.* ∎

COROLLARY 2. (i) *If $\kappa \geqq \beth_\omega$ is a Rowbottom cardinal then κ is a strong limit.*

(ii) *If $\kappa > \beth_{\omega_1}$ is a Rowbottom cardinal then κ is a fixed point of the beth function.* ∎

COROLLARY 3. *If κ is a Rowbottom cardinal greater than the first inaccessible then κ is the limit of the sequence of all inaccessible cardinals less than κ. In particular, if κ is regular then κ is the κth inaccessible cardinal.*

Similarly, if κ is a regular Rowbottom cardinal greater than the first Mahlo then κ is the κth Mahlo cardinal, etc. ∎

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